PROPERTIES OF MEROMORPHICALLY STARLIKE AND CONVEX FUNCTIONS

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ABSTRACT

In this paper i studied some properties of meromorphically starlike and meromorphically convex functions. We have proved $f(z) \in \tau_r^* [\alpha(p,q)]$ where $f(z) \in \Sigma_r$ satisfying an inequality given by

$$\sum_{n=0}^{\infty}\left(n+k+|2\alpha(p,q)+n-k|\right)|a_n|r^{n+1} \le 2[1-\alpha(p,q)]$$

In this paper I also proved that $f(z) \in C_r[\alpha(p,q)]$

where $f(z) \in \Sigma_r$ and satisfying the inequality given by

$$\sum_{n=1}^{\infty} n[n+\alpha(p,q)]|a_n|r^{n+1} \le 1-\alpha(p,q).$$

KEY WORDS: Univalent Function, Starlike Function, convex Function, Analytic Function.

1. INTRODUCTION:

Let Σ_r denote the class of functions f(z) of the form

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_n z^n$$

which are analytic in the disk $D_r=\{z\in c: 0<|z<r\le 1|\}$. A function $f(z)\in\sum_r$ is said to be strlike of order $\alpha(p,q)$ if it satisfies the inequality

$$\operatorname{Re}\left[-\frac{zf'(z)}{f(z)(p-q)}\right]>\alpha(p,q) \qquad (z\in D_r)$$

For some $\alpha(p,q)$ ($0\le\alpha(p,q)<1$). We say that f(z) is in the class $\tau_r^*(\alpha(p,q))$ for such functions. A function $f(z) \in \Sigma_r$ is said to be convex of order $\alpha(p,q)$ if it satisfies the inequality

$$\operatorname{Re}\left[-\left(1+\frac{zf''(z)}{f'(z)(p-q)}\right)\right]>\alpha(p,q) \qquad (z\in D_r)$$

For some $\alpha(p,q)$ ($0\le\alpha(p,q)<1$). We say that f(z) is in the class $C_r[\alpha(p,q)]$ if it is convex of order $\alpha(p,q)$ in D_r. We note that $f(z) \in C_r[\alpha(p,q)]$ if and only if $-z f(z) \in \tau^*_r[\alpha(p,q)]$. There are many papers discussing various properties of classes consisting of univalent, starlike, convex, multivalent, and meromorphic functions in the book by Srivastava and Owa.

Ozaki has shown that the necessary and sufficient condition that $f(z)\in\Sigma_r$ with $a_n \ge 0$ (n=1,2,3……) is meromorphic and univalent in D_r is that there should exist the relation

$$\sum_{n=1}^{\infty} na_n r^{n+1}\le 1$$

between its coefficients.

Cofficient Inequalities for functions

Theorem 1 : If $f(z) \in \Sigma_r$ satisfies for some $\alpha(p,q)$ ($0 \le \alpha(p,q) < 1$) and $k[\alpha(p,q) < k \le 1]$ then $f(z) \in \tau_r^*[\alpha(p,q)]$.

Proof: For $f(z) \in \Sigma_r$ we know that

$$|zf'(z)+kf(z)(p-q)|-|zf'(z)+[2\alpha(p,q)-k]f(z)(p-q)|$$

$$=\left|(k-1)\frac{1}{z}+\sum_{n=0}^{\infty}(n+k)a_n z^n\right|-\left|[2\alpha(p,q)-k-1]\frac{1}{z}+\sum_{n=0}^{\infty}[2\alpha(p,q)+n-k]a_n z^n\right|$$

By applying the condition of the theorem, we have

$$\le(k-1)+\sum_{n=0}^{\infty}(n+k)|a_n|r^{n+1}-(k+1-2\alpha(p,q))+\sum_{n=0}^{\infty}|2\alpha(p,q)+n-k||a_n|r^{n+1}$$

$$=2[\alpha(p,q)-1]+\sum_{n=0}^{\infty}(n+k+|2\alpha(p,q)+n-k|)|a_n|r^{n+1}$$

$$\le 0$$

Which shows that

$$\sum_{n=0}^{\infty}\left(n+k+|2\alpha(p,q)+n-k||a_n|r^{n+1}\right)\le 2[1-\alpha(p,q)]$$

It follows that

$$\left|\frac{zf'(z)+kf(z)(p-q)}{zf'(z)+[2\alpha(p,q)-k]f(z)(p-q)}\right|\le 1$$

So that $\operatorname{Re}\left(-\frac{zf'(z)}{f(z)(p-q)}\right)>\alpha(p,q) \qquad (z\in D_r)$

By Putting k=0, p=1 and q=0 in the above theoram. We have

Corollary –1 : If f(z)∈ $\sum_r$ satisfies

$$\sum_{n=0}^{\infty}(n+\alpha)|a_n|r^{n+1}\le 1-\alpha$$

For some α $[1/2\le\alpha<1]$,then f(z)∈$\tau^*_r(\alpha)$.

Putting p=1 & q=0, we have

Corollary –2 : Let the function f(z)∈$\sum_r$ be given in the introduction with $a_n=|a_n|e^{\frac{-n+1}{2\pi}i}$,

then f(z) ∈ $\tau^*_r(\alpha)$ if and only if

$$\sum_{n=0}^{\infty}(n+\alpha)|a_n|r^{n+1}\le 1-\alpha$$

For some α $[1/2\le\alpha<1]$

Proof: In view of the above theorem, we see that if the coefficient inequality holds true for some α $[1/2\le\alpha<1]$, then f(z)∈$\tau^*_r(\alpha)$.
Conversely, let f(z) be in the class $\tau^*_r(\alpha)$, then

$$\operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right)=\operatorname{Re}\left(\frac{1-\sum_{n=0}^{\infty}na_nz^{n+1}}{1+\sum_{n=0}^{\infty}a_nz^{n+1}}\right)>\alpha$$

For all z ∈ D_r.

Letting z= r $e^{1/2\Pi i}$, we have that $a_n z^{n+1} = |a_n| r^{n+1}$
which implies that

$$1-\sum_{n=0}^{\infty}n|a_n|r^{n+1}\ge\alpha\left(1+\sum_{n=0}^{\infty}|a_n|r^{n+1}\right).$$

which is equivalent to

$$\sum_{n=0}^{\infty}(n+\alpha)|a_n|r^{n+1}\le 1-\alpha$$

Example : The function f(z) given by

$$f(z)=\frac{1}{z(p-q)}+a_0+\left(\frac{1-\alpha(p,q)-\alpha(p,q)|a_0|}{n+\alpha(p,q)}\right)e^{i\theta}z^n$$

belongs to the class $\tau^*_r(\alpha)$ for some real 'θ' with ½ ≤ α (p,q) ≤ $\frac{1}{1+|a_0|}$ < 1.

Corollary 3 : Let the function f(z) ∈ $\sum_r$ be given by introduction with $a_n \ge 0$, then

f(z) ∈ $\tau^*_r[\alpha(p,q)]$ if and only if

$$\sum_{n=0}^{\infty}[n+\alpha(p,q)]a_nr^{n+1}\le 1-\alpha(p,q)$$

for some α (p,q), $[1/2\le\alpha(p,q)<1]$.

Remark 1: If f(z) ∈ $\sum_r$ with $a_{0=0}$, then corollalry-2 holds true for some α (p,q) [0 ≤ α (p,q) < 1].

Remark 2: If f(z) ∈ $\sum_r$ with $a_0 = 0$, then corollary-3 holds true for 0 ≤ α (p,q) < 1.

Theorem 2 : If f(z) ∈ $\sum_r$ satisfies

$$\sum_{n=1}^{\infty}n[n+\alpha(p,q)]|a_n|r^{n+1}\le 1-\alpha(p,q)$$

for some α (p,q) [0 ≤ α (p,q) < 1], then f(z) belongs to the class $C_r[\alpha(p,q)]$.
Proof : Note that f(z) ∈ $C_r[\alpha(p,q)]$ if and only if

–z f' (z) ∈ $\tau^*_r[\alpha(p,q)]$, and

$$-zf'(z)=\frac{1}{z(p-q)}-\sum_{n=1}^{\infty}na_nz^n,$$

With the help of theoram-1, we complete the proof of the theorem.

Example 2 : The function f(z) given by

$$f(z)=\frac{1}{z(p-q)}+a_0+\left[\frac{1-\alpha(p,q)}{n(n+\alpha(p,q))}\right]e^{i\theta}z^n$$

belongs to the class Cr [α (p,q)] for some real θ with 0 ≤ α (p,q) < 1.

Corollary 4 : Let the function f(z) ∈ $\sum_r$ be given by introduction with $a_n=|a_n|e^{\frac{-n+1}{2\pi}i}$,

then f(z) ∈ C_r α (p,q) if and only if the inequality (remark -1) holds true for some α (p,q) , [0 ≤ α (p,q)<1].

Corollary 5 : Let the function $f(z)\in\sum_r$ be given by introduction with $a_n \geq 0$, then $f(z)\in C_r[\alpha(p,q)]$ if and only if

$$\sum_{n=1}^{\infty} n[n+\alpha(p,q)]a_n r^{n+1} \leq 1-\alpha(p,q)$$

for some α (p,q) [$0 \leq \alpha(p,q) < 1$].

Starlikeness and convexity of functions

Consider the radius problems for starlikeness and convexity of functions f(z) belonging to the class $\sum_r$.

Theorem:3 A function $f(z)\in\sum_r$ belongs to the class $\tau^*_r[\alpha(p,q)]$ for $0 \leq r \leq r_0$ where r_0 is the smallest positive root of the equation.

$$\alpha(p,q)|a_0|r^3-(\delta+1-\alpha(p,q))r^2-\alpha(p,q)|a_0|r+1-\alpha(p,q)=0$$

And

$$\delta=\sqrt{\sum_{n=1}^{\infty} n|a_n|^2}+\alpha(p,q)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n}|a_n|^2}$$

Proof:_ Using the Cauchy inequality, we have

$$\sum_{n=1}^{\infty}[n+\alpha(p,q)]|a_n|r^{n+1}=\alpha(p,q)|a_0|r+\sum_{n=1}^{\infty}|a_n|r^{n+1}$$

$$\leq \alpha(p,q)|a_0|r+\sqrt{\sum_{n=1}^{\infty} n|a_n|^2}\sqrt{\sum_{n=1}^{\infty} nr^{2n+2}}+\alpha(p,q)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n}|a_n|^2}\sqrt{\sum_{n=1}^{\infty} nr^{2n+2}}$$

$$=\alpha(p,q)|a_0|r+\sqrt{\frac{r^4}{(1-r^2)^2}}\left(\sqrt{\sum_{n=1}^{\infty} n|a_n|^2}+\alpha(p,q)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n}|a_n|^2}\right)$$

$$=\alpha(p,q)|a_0|r+\frac{r^2}{1-r^2}\delta$$

$$<1-\alpha(p,q)$$

By corollary - 1 we get $f(z) \in \tau^*_r[\alpha(p,q)]$ for $0 \leq r \leq r_0$ Letting $a_0=0$, p=1, q=0 in the above theorem.

corollary : 6 A function $f(z) \in \sum_r$ with $a_0=0$ belongs to the class $\tau^*_r(\alpha)$ for $0 \leq r < r_0$.

$$r_0=\sqrt{1-\frac{\delta}{\delta+1-\alpha}}$$

Where $\delta=\sqrt{\sum_{n=1}^{\infty} n|a_n|^2}+\alpha\sqrt{\sum_{n=1}^{\infty}\frac{1}{n}|a_n|^2}$

Example 3:_ If we consider the function f(z) given by

$$f(z)=\frac{1}{z(p-q)}+\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}e^{i\theta_n}z^n \qquad (\theta_n \text{ is real})$$

then $f(z) \in \tau^*_r[\alpha(p,q)]$ for $0 \leq r < r_0$ with

$$\delta=\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}+\alpha(p,q)\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^4}}$$

$$=\sqrt{\zeta(2)}+\alpha(p,q)\sqrt{\zeta(1)}$$

$$=\pi\left(\frac{1}{\sqrt{6}}+\frac{\pi\alpha(p,q)}{3\sqrt{10}}\right)$$

By Putting $\alpha=0$ we have

$$\delta=\frac{\pi}{\sqrt{6}} \cong 1.282550$$

and $r_0=\sqrt{\frac{\sqrt{6}}{\sqrt{6}+\pi}} \cong 0.661896.$

Theorem 4 : A function $f(z) \in \sum_r$ belongs to the class $C_r[\alpha(p,q)]$ for $0 \leq r < r_1$ where

$$r_1=\sqrt{1-\frac{\sigma}{\sigma+1-\alpha(p,q)}}$$

Where $\sigma=\sqrt{\sum_{n=1}^{\infty} n^3|a_n|^2}+\alpha(p,q)\sqrt{\sum_{n=1}^{\infty} n|a_n|^2}$

Example :_ Let us consider the function f(z)

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{1}{n^2\sqrt{n}}e^{i\theta_n}z^n \qquad (\theta_n \text{ is real})$$

see that $f(z) \in C_r[\alpha(p,q)]$ for $0 \leq r < r_0$ with $\delta=\pi\left(\frac{1}{\sqrt{6}}+\frac{\pi\alpha}{3\sqrt{10}}\right)$

Taking $\alpha=0$,

we get $\delta=\frac{\pi}{\sqrt{6}} \cong 1.282550$

and $r_0=\sqrt{\frac{\sqrt{6}}{\sqrt{6}+\pi}} \cong 0.661896.$

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